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Quantum Theory As a Theory in a Classical Propositional Calculus

Othman Qasim Malhas[1]

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Classical logic and Boolean algebras are, of course, very intimately related. It is, however, possible to show that lattices of propositions isomorphic to the lattice of all the closed subspaces of a separable Hilbert space arise quite naturally within the classical propositional logic. This was first shown by the author in 1987 in connection with a certain type of theories called *theories with orthocomplementation*. These theories are not easy to interpret physically and it is shown that simpler theories, which are more amenable to physical interpretation, can also be used. It is then possible to assume that quantum theory is such a theory and, as a result, to formulate a new approach that provides a way of looking at the wave–particle duality and touches upon the foundations of quantum field theory.

1. THE POSET OF A THEORY

To begin with, let us establish some terminology. Let U be a nonempty set of symbols not containing the symbols $\rightarrow$, $\sim$, (,). The cardinality of U is strictly greater than 0, otherwise it is arbitrary. By *the classical propositional calculus generated by* U we mean the smallest set W such that

1. $U \subseteq W$.
2. If $A \in W$, then $(\sim A) \in W$.
3. If $A, B \in W$, then $(A \rightarrow B) \in W$.

We may remove brackets where there is no danger of ambiguity. *Logical connectives* other than $\sim$ and $\rightarrow$ can be defined in the usual way. For example, we may consider $A \vee B$ as an abbreviation for $\sim A \rightarrow B$; $A \wedge B$ as an abbreviation for $\sim(A \rightarrow \sim B)$; and $A \leftrightarrow B$ as an abbreviation for $(A \rightarrow B) \wedge (B \rightarrow A)$. The elements of U may be called *simple propositions* and those of $W \backslash U$ *compound propositions*. A *valuation* of W is a function

[1]Department of Mathematics, Yarmouk University, Irbid, Jordan.

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$t: W \to \{0, 1\}$ such that

$$t(A) \neq t(\sim A)$$

and

$$t(A \to B) = 0 \quad \text{iff} \quad t(A) = 1 \quad \text{and} \quad t(B) = 0$$

If $A \in W$ and t is a valuation, then $t(A)$ is the *truth value* of A under t. A function $s: U \to \{0, 1\}$ shall be called *an assignment to* U. It is very important to emphasize that *any assignment to* U *can be extended to a unique valuation of* W. This assertion, which guarantees the existence of valuations, is an obvious consequence of the fact that the truth value of a compound proposition is uniquely determined by the truth values of the simple propositions which occur in it.

Now, $A \in W$ is a *tautology* iff $t(A) = 1$ for every valuation t. If $K \subseteq W$ and there exists a valuation t such that $t(K) = 1$, i.e., the truth value of every element of K under t is 1, then K is *consistent*. If $K \subseteq W$ and $A \in W$ and $t(A) = 1$ whenever $t(K) = 1$, then we say that A is a *logical consequence of* K. A set $K \subseteq W$ is a *theory* in W iff every logical consequence of K belongs to K. Theories exist because if Ω is a nonempty set of valuations, then the set $T(\Omega) = \{A \in W: t(A) = 1 \text{ for every } t \in \Omega\}$ is easily seen to be a theory. Furthermore, this theory is consistent because Ω is nonempty. We may call $T(\Omega)$ *the theory of* Ω.

Partially ordered sets of (classes of) propositions, which are not necessarily Boolean algebras, arise by examining the influence of theories in W on U, the set of all simple propositions. Let T be a theory in W. For all $p, q \in U$ put $p \equiv q$ iff $(p \leftrightarrow q) \in T$. Clearly, $\equiv$ is an equivalence relation. For every $p \in U$, let $[p]$ be the equivalence class of p and let U_T be the collection of equivalence classes. U_T is partially ordered by

$$[p] \leq [q] \quad \text{iff} \quad (p \to q) \in T$$

The partially ordered set $(U_T, \leq)$ shall be called *the poset of* T. For all $p, q \in U$, the l.u.b. and g.l.b. of $\{[p], [q]\}$ need not exist in U_T. If they exist, then they shall respectively be denoted by $[p] \oplus [q]$ and $[p] \otimes [q]$.

It is now easy to see that $(U_T, \leq)$ is not necessarily a Boolean algebra. Let T, for example, be the set of all logical consequences of p, for some fixed $p \in U$. An obvious truth table argument will confirm that whereas $[p]$ is a maximum element in U_T, there does not exist $r \in U$ such that $r \to q \in T$, for every $q \in U$. Hence $(U_T, \leq)$, unlike a Boolean algebra, has no minimum. Other choices of T produce other deviations from Boolean algebras.

2. f-VALUATIONS

The question arises whether there is a theory T for which $(U_T, \leq)$ is isomorphic to a given partially ordered set, e.g., the lattice of all the closed subspaces of a separable Hilbert space. We shall see that there is. To that end let $(L, \leq)$ be any atomic orthocomplemented lattice with the following property.

D: For all $a, b \in L$, b dominates a (i.e., $a \leq b$) iff every atom dominated by a is dominated by b.

The lattice $L(H)$ of all the closed subspaces of a separable Hilbert space H has property **D** and so does any atomic Boolean algebra. Let $\mathbf{o}$, $\mathbf{i}$ be the null and unit elements of L. The orthogonal complement of $a \in L$ is denoted $a^{\perp}$. Elements a, b of L are said to be orthogonal iff $a \leq b^{\perp}$ and then we write $a \perp b$.

Next choose U so that the cardinality of U is greater than that of L. Thus, there is a surjection f from U onto L. For every atom $e \in L$ let $s_e : U \to \{0, 1\}$ be defined by

$$s_e(p) = 1 \qquad \text{iff} \quad e \leq f(p)$$

As has already been emphasized, the assignment s_e can be extended to a unique valuation t_e of W that we shall call an *f-valuation*. Once the function f has been chosen, let it remain fixed. Different f-valuations, for our fixed f, are obtained by choosing different atoms $e \in L$. Let Ω be the set of all f-valuations and let T be the theory of Ω, i.e., $T = T(\Omega)$. (See Section 1 above.) Then T is consistent because Ω is nonempty. Since f is a surjection, there exist $i, o \in U$ such that $f(i) = \mathbf{i}$ and $f(o) = \mathbf{o}$.

Theorem 1. $i \in T$ and $\sim o \in T$.

Proof. Since $f(i) = \mathbf{i}$, we have $e \leq f(i)$ for every atom $e \in L$. Hence i takes value 1 in every f-valuation. Thus, $i \in T$. On the other hand, $f(o) = \mathbf{o}$ and for any atom $e \in L$, $e \nleq f(o)$. Thus o is false in every f-valuation, i.e., $(\sim o)$ takes value 1 in every f-valuation. Hence $(\sim o) \in T$. ∎

Theorem 2. For all $p, q \in U$, $(p \to q) \in T$ iff $f(p) \leq f(q)$.

Proof. First suppose that $f(p) \nleq f(q)$. Then by property **D**, there is an atom $e \in L$ such that $f(p)$ dominates e, but $f(q)$ does not dominate e. Then, for the f-valuation t_e we have $t_e(p) = 1$ and $t_e(q) = 0$. Thus, $t_e(p \to q) = 0$. Hence $(p \to q) \notin T$.

Conversely, suppose $(p \to q) \notin T$. Then there exists an f-valuation t_e such that $t_e(p) = 1$ and $t_e(q) = 0$. Thus $e \leq f(p)$, but $e \nleq f(q)$. By property **D**, $f(p) \nleq f(q)$. ∎

Corollary. (i) For all $p, q \in U, f(p) \leq f(q)$ iff $[p] \leq [q]$ and $f(p) = f(q)$ iff $[p] = [q]$ and (ii) for all $p \in U$, $[o] \leq [p] \leq [i]$.

Proof. Part (i) follows directly from the definition of the partial order relation in U_T, the equivalence relation on U, and Theorem 2. For part (ii) we note that for all $p \in U$, $\mathbf{o} \leq f(p) \leq \mathbf{i}$ or, equivalently, $f(o) \leq f(p) \leq f(i)$. By part (i), $[o] \leq [p] \leq [i]$. ■

For all $p, q \in U$ we say that q is a *conjugate* of p iff $f(q) = (f(p))^{\perp}$. Clearly, this is a symmetric relation. For all $p, q \in U$, if q is a conjugate of p, we put $[q] = [p]^*$. That is, we define $[p]^*$ to be the equivalence class of a conjugate of p. By the corollary of Theorem 2, the operation * is well defined.

Theorem 3. The operation * is an orthocomplementation of U_T. That is, for all $u, v \in U_T$:

(i) $u^{**} = u$.
(ii) $u \otimes u^*$ and $u \oplus u^*$ exist and respectively are $[o]$ and $[i]$.
(iii) $u \leq v$ implies $v^* \leq u^*$.

Proof. (i) Suppose $u = [p]$. Then $u^* = [p]^* = [q]$, where q is a conjugate of p. Thus $u^{**} = [p]^{**} = [q]^*$. But since p is a conjugate of q (the relation "*is a conjugate of*" is symmetric), we have $[q]^* = [p]$. Thus, $u^{**} = [p] = u$.

To prove (ii), again let $u = [p]$ and q be a conjugate of p. Then, again, $[q] = [p]^*$. But $f(p) \leq f(i)$ and $f(q) \leq f(i)$. Then by the corollary of Theorem 2, $[p] \leq [i]$ and $[q] = [p]^* \leq [i]$. Thus $[i]$ is an upper bound of u and u^*. If $v = [r]$ is another upper bound, then, by the corollary of Theorem 2, $f(r)$ is an upper bound of $f(p)$ and $f(q) = f(p)^{\perp}$. Thus $f(i) \leq f(r)$, since $f(i)$ is the least upper bound of $f(p)$ and $f(p)^{\perp}$. Thus, by the corollary of Theorem 2, $[i] \leq [r]$. That is, $[i]$ is the least upper bound of $u = [p]$ and $u^* = [q]$. Similarly, $[o]$ is the g.l.b. of $\{u, u^*\}$.

To prove (iii), let $u = [p]$ and $v = [r]$ and suppose $u \leq v$. Then $f(p) \leq f(r)$ and we have $f(r)^{\perp} \leq f(p)^{\perp}$. Further suppose that q is a conjugate of p, and w is a conjugate of r. Then $f(r)^{\perp} = f(w)$ and $f(p)^{\perp} = f(q)$. Thus $f(w) \leq f(q)$. Hence, by the corollary of Theorem 2, $[w] \leq [q]$, i.e., $[r]^* \leq [p]^*$, i.e., $v^* \leq u^*$. ■

Thus, $(U_T, [o], [i], \leq, {}^*)$ is, like L, an *orthocomplemented partially ordered set* (*orthoposet* for short). The next theorem shows that this orthoposet is also a lattice isomorphic to L.

Theorem 4. The orthoposets U_T and L are isomorphic.

Proof. Let $h: U_T \to L$ be given by $h([p]) = f(p)$. Then h is a bijection and, by the corollary of Theorem 2,

$$[p] \leq [q] \quad \text{iff} \quad h([p]) \leq h([q])$$

Also

$$h([o]) = f(o) = \mathbf{o} \qquad \text{and} \qquad h([i]) = f(i) = \mathbf{i}$$

Finally,

$$h([p]^*) = h([q])$$

where q is a conjugate of p. Thus $h([p]^*) = f(q) = f(p)^{\perp}$. ■

Theorem 4 is the central result of this paper. It tells us that the orthoposet of T "faithfully mimics" the orthoposet L. In particular, if L is an atomic Boolean algebra, then so is U_T. If L is the lattice of all the closed subspaces of a separable Hilbert space, then U_T is isomorphic to it. Thus, a nondistributive lattice of propositions does not necessarily require the formulation of a new logic. The possibility exists that we may be able to consider the lattice of all "quantum propositions" as the poset of a theory in a classical propositional calculus. This possibility shall be explored further in Section 6, below. In the meantime we must clarify the relationship between logical negation and orthocomplementation in U_T.

Theorem 5. For all $p, q \in U$, $f(p) \perp f(q)$ implies $(p \to \sim q) \in T$.

Proof. Suppose that $(p \to \sim q) \notin T$. Then there is an f-valuation t_e such that

$$t_e(p) = 1 \qquad \text{and} \qquad t_e(q) = 1$$

Thus $e \leq f(p)$ and $e \leq f(q)$ and $e \neq \mathbf{o}$, because e is an atom. Hence $f(p)$ is not orthogonal to $f(q)$. ■

To put it somewhat loosely, this says that two simple propositions represented by two orthogonal elements of L cannot simultaneously have truth value 1 under T. The converse of Theorem 5, however, is not necessarily correct. To see this, let L be the lattice of all the subspaces of the Euclidean space $\mathfrak{R}^3$. Let p be a simple proposition (an element of U) such that $f(p)$ is the xy plane and let q be a simple proposition such that $f(q)$ is the one-dimensional subspace generated by the vector $(1, 1, 1)$. Then $f(p)$ is not orthogonal to $f(q)$ and yet $(p \to \sim q) \in T$, since no atom, i.e., a one-dimensional subspace, is contained in both $f(p)$ and $f(q)$ and, consequently, there does not exist an f-valuation in which both p and q have truth values 1. For a *theory with orthocomplementation* both Theorem 5 and its converse are true (Malhas, 1987). Thus, T is not such a theory.

If p and q are conjugate propositions, then $f(p) = f(q)^{\perp}$. Then not only are $f(p)$ and $f(q)$ orthogonal, which implies $(p \to \sim q) \in T$, but also one is the *orthogonal complement* of the other. For this particular case one may be tempted to anticipate that this should imply that q is equivalent under T to the *negation* of p. More precisely, that *if q is a conjugate of p, then* $(p \leftrightarrow \sim q) \in T$. This, however, is not so: Let L be the lattice of all the subspaces of the Euclidean space $\mathfrak{R}^3$, as above and let e be the one-dimensional subspace generated by the vector (1, 1, 1). Let $p, q \in U$ be such that $f(p)$ is the xy plane and $f(q)$ is the z axis. Then q is a conjugate of p, but $t_e(p) = t_e(q) = 0$, which implies $t_e(p \leftrightarrow \sim q) = 0$, i.e., $(p \leftrightarrow \sim q) \notin T$. More shall be said about the meaning of the orthocomplementation in U_T in Section 4 on occupation sets.

3. THE LINDENBAUM ALGEBRA OF T

It is a great relief that orthocomplementation does not correspond to logical negation. Similarly, we should not, in general, expect the least upper and greatest lower bounds of two elements in U_T, where they exist, to "correspond," in any naive sense, to the logical connectives $\wedge$, and $\vee$. The reason is simple: We must not run foul of the Kochen and Specker (1967) theorem, or other variations on the same theme, e.g., Zierler and Schlessinger (1965): *We cannot embed the lattice of all the closed subspaces of a separable Hilbert space into any Boolean algebra.* In particular, we cannot embed the orthocomplemented lattice U_T into the "Lindenbaum algebra" of T. This latter structure is obtained as follows:

Define an equivalence relation $\approx$ on W (the set of all, simple and compound, propositions) by setting $A \approx B$ iff $(A \leftrightarrow B) \in T$. Let W_T be the set of all equivalence classes. The equivalence class of A shall be denoted by $[[A]]$. Partially order W_T by setting $[[A]] \angle [[B]]$ if $(A \rightarrow B) \in T$ and set $[[A]]^\dagger = [[\sim A]]$. It turns out that

$$(W_T, \angle, ^\dagger)$$

is a Boolean algebra, with † being the operation of complementation. The unit element $\mathbf{I}$ is the equivalence class of any consequence of T, e.g., any tautology. The null element $\mathbf{0}$ is the equivalence class of the negation of any consequence of T, e.g., the negation of a tautology. By Theorem 1, we have

$$\mathbf{0} = [[o]] = [[\sim i]]$$

and

$$\mathbf{I} = [[i]] = [[\sim o]]$$

An obvious truth table argument (or other elementary methods) will demonstrate that for all $A, B, C \in W$, if $A \rightarrow C \in T$ and $B \rightarrow C \in T$, then $(A \vee B) \rightarrow C \in T$. It easily follows that for all $[[A]], [[B]] \in W_T$

$$\text{l.u.b. of } \{[[A]], [[B]]\} = [[A \vee B]]$$

Similarly,

$$\text{g.l.b. of } \{[[A]], [[B]]\} = [[A \wedge B]]$$

We shall call the Boolean algebra $(W_T, \angle, ^\dagger)$ *the Lindenbaum algebra* of T.

From the definitions it follows that for all $A \in W$ and $p \in U$, $A \in [p]$ implies $A \in U$ and $A \leftrightarrow p \in T$, which implies that $A \in [[p]]$. Thus, for every $p \in U$, $[p] \subseteq [[p]]$. Define ζ: $U_T \to W_T$ by putting $\zeta([p]) = [[p]]$. It immediately follows that ζ is injective (but not surjective). It also follows that for all $p, q \in U$,

$$[p] \leq [q] \quad \text{iff} \quad \zeta([p]) \angle \zeta([q])$$

$$\zeta([i]) = \mathbf{I} \quad \text{and} \quad \zeta([o]) = \mathbf{0}$$

Thus, ζ is an embedding of the partially ordered set $(U_T, \leq)$ into the partially ordered set $(W_T, \angle)$. (There is no need to suspect that something has gone wrong here. This embedding exists, because we have just shown that it does and, hence, it does not contravene any of the "no embedding" theorems. This embedding is an embedding of U_T into W_T as *partially ordered sets* and not as *orthocomplemented* lattices.) Without loss of rigor we may therefore identify the elements of U_T with elements of W_T. To be precise, for all $p \in U$, we identify $[p]$ with $[[p]] = \zeta([p])$. Thus we may consider U_T to be a subset of W_T. The element of W_T corresponding to $[p]^*$ shall simply be denoted by $[[p]]^*$. That is, for all $p, q \in U$, if p is a conjugate of q, then $\zeta([q]) = \zeta([p]^*)$ and we define $[[p]]^* = \zeta([p]^*)$.

When this identification is made, it becomes obvious that the *least* upper bound in U_T, *if it exists*, of two elements in U_T is an *upper* bound of these two elements in W_T and must, therefore, be "higher" than their *least* upper bound in W_T. More precisely, for all $p, q \in U$

$$[[p \vee q]] \angle [[p]] \oplus [[q]]$$

Similarly,

$$[[p]] \otimes [[q]] \angle [[p \wedge q]]$$

For all $p, q \in U$, if p is a conjugate of q, then $f(p) = f(q)^\perp$ and, by Theorem 5, we have $p \to \sim q \in T$. Then, directly from the definitions, for all $p, q \in U$. If p is a conjugate of q, then $p \to \sim q \in T$ and we have $[[p]] \angle [[\sim q]]$ or, equivalently, $[[p]] \angle [[q]]^\dagger$ or, since $[[p]] = [[q]]^*$,

$$[[q]]^* \angle [[q]]^\dagger$$

But we have already seen that since it is not necessarily the case that $(p \leftrightarrow \sim q) \in T$, when p and q are conjugate, then it is *not* necessarily the case that $[[p]] = [[\sim q]]$ or, equivalently, that $[[q]]^* = [[\sim q]]$ or, also equivalently, that $[[q]]^* = [[q]]^\dagger$.

Conclusion: U_T is a subset of W_T and it is partially ordered by the same partial order relation $\angle$ in W_T, i.e., the restriction of $\angle$ to U_T, but insofar as U_T is orthocomplemented, then the operation * of orthocomplementation in U_T is not the restriction of the orthocomplementation † in W_T to U_T. The most that can be said is that for all $u \in W_T$, $u^* \angle u^\dagger$. Similarly, if U_T is a lattice, then the l.u.b. of any two elements in U_T is an upper bound of the two elements as elements in W_T and their g.l.b. in U_T is a lower bound of them in W_T.

4. OCCUPATION SETS

Let $\mathfrak{R}$ be the real line and let $\mathbf{B}$ be the σ-field of Borel sets on $\mathfrak{R}$. Let f be a surjection from U to L, where L now is the lattice of all closed subspaces of a separable Hilbert space H. (The partial order relation in L is now *set inclusion.*) Let T be the theory of the set Ω of all f-valuations. Then, as we have seen, U_T is isomorphic to L. Define an *L-valued measure on* $\mathfrak{R}$ to be a function $M: \mathbf{B} \to L$ satisfying the following conditions.

1. $M(\varnothing) = \mathbf{o}$ and $M(\mathfrak{R}) = \mathbf{i}$.
2. If E, F are disjoint Borel sets, then $M(E) \perp M(F)$.
3. If $E_1, E_2, E_3, \ldots$ is a sequence of pairwise disjoint Borel sets, then $M(\cup E_i) = \sum M(E_i)$.

The $\sum$ in the last condition denotes *lattice sum*, i.e., the operation of taking the least upper bound of a countable collection of elements in a σ-lattice. L-valued measures exist: For each nonzero $a \in L$, let N_a be the function from $\mathbf{B}$ to L defined by

$$N_a(E) = \begin{cases} a & \text{if } \ 0 \notin E \text{ and } 1 \in E \\ a^\perp & \text{if } \ 0 \in E \text{ and } 1 \notin E \\ \mathbf{i} & \text{if } \ 0 \in E \text{ and } 1 \in E \\ \mathbf{o} & \text{if } \ 0 \notin E \text{ and } 1 \notin E \end{cases}$$

It is easy to verify that N_a is an L-valued measure. It is also important to note that, in general, if E is a Borel set and E' is the complement of E in $\mathfrak{R}$, and M is an L-valued measure, then

$$M(E') = M(E)^\perp$$

By the spectral theorem, there is a natural 1–1 correspondence between the self-adjoint operators on H and L-valued measures. See Jauch (1968), for instance.

Definition. For every L-valued measure M and any (fixed) atom $e \in L$, let $C_M(e)$ be the intersection of all the Borel sets E such that $e \subseteq M(E)$. Then $C_M(e)$ shall be called *the occupation set for M in the f-valuation t_e.*

If our Hilbert space is of a finite number of dimensions, then each L-valued measure is represented by a self-adjoint operator (i.e., Hermitian matrix). If M is such an operator, then $M(E)$ is the subspace generated by the eigenspaces of all the eigenvalues of (the matrix corresponding to) M that happened to be in E. If e is an atom, i.e., one-dimensional subspace, then $C_M(e)$ is the smallest set of eigenvalues of M such that e is contained in the span of the corresponding eigenvectors. Thus, for example, if our Hilbert space is the 3-dimensional Euclidean space $\mathfrak{R}^3$ and if M is represented by

$$\begin{pmatrix} 100 \\ 020 \\ 003 \end{pmatrix}$$

and e is the one-dimensional subspace generated by the vector $(1, 1, 1)$, then $C_M(e) = \{1, 2, 3\}$. If e is the one-dimensional subspace generated by the vectors $(1, 1, 0)$, then $C_M(e) = \{1, 2\}$. If e is the x axis, then $C_M(e) = \{1\}$ and so on. *It is important to note that in general $C_M(e)$ need not be a singleton.*

Theorem 6. For each L-valued measure M and each atom $e \in L$, $C_M(e) \neq \varnothing$, and if $C_M(e) \subseteq E$, for some Borel set E, then $e \subseteq M(E)$.

Proof. By the spectral theorem there exists a measure μ on $\mathfrak{R}$ (more accurately, on the *spectrum* of M) such that we may identify the abstract Hilbert space H with $L^2(\mu)$. For every Borel set E, the closed subspace $M(E)$ is the set of all $\eta \in L^2(\mu)$ which take the value 0 outside E. The atom e is generated by $\psi \in L^2(\mu)$ of norm 1. Let F be the set of all real numbers x for which $\psi(x) \neq 0$. The condition $e \subseteq M(E)$ holds iff $F \subseteq E$. The intersection of all the Borel sets E such that $e \subseteq M(E)$ contains F. Now F is nonempty, for otherwise we would have $e \subseteq M(\varnothing) = \mathbf{o}$, which contradicts the fact that e is an atom. ■

The importance of occupation sets is a consequence of the following theorem, which relates occupation sets to assignments of truth values. We shall use this theorem to explain why orthocomplementation in U_T does not correspond to negation.

Theorem 7. For each $p \in U$ and each atom $e \in L$, if $f(p) = M(E)$ for some L-valued measure M and some Borel set E, then $t_e(p) = 1$ iff $C_M(e) \subseteq E$.

Proof. First suppose that $t_e(p)=1$. Then $e \subseteq M(E)$. Hence $C_M(e) \subseteq E$, by the definition of $C_M(e)$. Conversely, suppose that $C_M(e) \subseteq E$. Then by Theorem 6, $e \subseteq M(E)$. Thus $t_e(p)=1$. ■

The concept of "truth" introduced in Theorem 7 provides intuitive insight into the relation of a proposition $p \in U$ to a conjugate $q \in U$. For example, suppose $f(p)=M(E)$. If q is a conjugate of p, then $f(q)=M(E')$. Hence if, for some atom $e \in L$, $C_M(e)$ is partly in E and partly in E' [i.e., if $E \cap C_M(e) \neq \varnothing$ and $E' \cap C_M(e) \neq \varnothing$], then both $t_e(p)=0$ and $t_e(q)=0$. Then neither $C_M(e) \subseteq E$ nor $C_M(e) \subseteq E'$. Thus p and q can simultaneously have truth value 0. The propositions p and q cannot, however, simultaneously have truth value 1, because $C_M(e)$ cannot simultaneously be a subset of E and of E'.

5. RELATION TO THREE-VALUED LOGICS

The concept of "truth" introduced in the last section can be related to many-valued logic in the following way: For every f-valuation t_e we introduce a function $\bar{\omega}_e$ from U onto the set $\{0, 0.5, 1\}$ as follows: Let $e \in L$ be an atom. For all $p \in U$, suppose that $f(p)=M(E)$, for some L-valued measure M and Borel set E. Then define

$$\bar{\omega}_e(p)=\begin{cases} 0.5 & \text{if} \quad C_M(e) \cap E \neq \varnothing \text{ and } C_M(e) \cap E' \neq \varnothing \\ 1 & \text{if} \quad C_M(e) \subseteq E \\ 0 & \text{if} \quad C_M(e) \subseteq E' \end{cases}$$

We can think of $\bar{\omega}_e(p)$ as a measure of *the degree of truth* of p in the valuation t_e. In fact, suppose $f(p)=M(E)$ as above. Then

$$t_e(p)=1 \quad \text{iff} \quad \bar{\omega}_e(p)=1$$

This follows immediately from the definition of $\bar{\omega}_e(p)$ above and from the fact that $t_e(p)=1$ iff $C_M(e) \subseteq E$. See Theorem 7. It follows that

$$t_e(p)=0 \quad \text{iff} \quad \bar{\omega}_e(p)<1$$

We may say that $p \in U$ *is certainly true* with respect to $\bar{\omega}_e$ if $\bar{\omega}_e(p)=1$, i.e., if the degree of truth of p is 1, and that $p \in U$ *is certainly false* if $\bar{\omega}_e(p)=0$, i.e., if the degree of truth of p is 0. Thus, f-valuations provide coarse estimates of truth: $t_e(p)=1$, i.e., p has *truth value* 1 in the f-valuation t_e means that p is certainly true and $t_e(p)=0$ says that p is not certainly true.

Now we can, if we so wish, extend $\bar{\omega}_e$ to a *generalized valuation* ω_e: $W \rightarrow \{0, 0.5, 1\}$ in any *reasonable* way, i.e., so that if $\omega_e(A)=t_e(A)=1$

or 0, then $\omega_e(\sim A)=t_e(\sim A)$, but if $\omega_e(A)=t_e(A)$ and $\omega_e(B)=t_e(B)$, then $\omega_e(A\rightarrow B)=t_e(A\rightarrow B)$.

For example, we may agree that for all $A\in W$, $\omega_e(\sim A)=1-\omega_e(A)$ and that $\omega_e(A\rightarrow B)=1$ if $\omega_e(A)\leq 0.5$ or $\omega_\varepsilon(B)\geq 0.5$ and $\omega_e(A\rightarrow B)=0$ otherwise. Needless to say, there are other reasonable ways of extending $\bar{\omega}$ to a function from W into $\{0, 0.5, 1\}$.

6. PHYSICAL APPLICATION

For the purposes of physical application we may think of U as a set of "simple" propositions in some field of empirical investigation. Clearly, empirical propositions are somewhat "fuzzy." The truth value of such a proposition is not always well defined. For our purposes truth value 1 signifies *certainly true* and truth value 0 signifies *not certainly true*.

The set W will then consist of U together with the set of all propositions that can be obtained from U by applying logical connectives in the usual way. Whether a proposition, i.e., an element of W, is certainly true or not will, in general, depend on time. An element of W may at one time be certainly true, but at other times not certainly true. Define a *valuation* to be a function $t: W\rightarrow\{0, 1\}$ satisfying the usual truth table rules of the classical propositional calculus (as in the definition of *a valuation* in Section 1), with the understanding that $t(A)=1$ means that A is certainly true and that $t(A)=0$ means that A is not certainly true.

Not every valuation, however, is allowed by nature. If Ω is the set of all allowed valuations, then $T(\Omega)$, the theory of Ω, is the set of all propositions which are certainly true, i.e., take truth value 1, in all allowed valuations. Thus $T(\Omega)$ is the set of "laws of nature" governing the given field of investigation. $T(\Omega)$ consists of all those assertions which are *always* certainly true. Here, "always" means "in every allowed valuation" rather than "for all moments of time." If, however, we agree that only one allowed valuation is "active" at any time, then it follows that "the laws of nature," as envisaged here, do not change with time.

To be more specific, let us take U to be the set of all English statements of the form

$$Q \text{ has a value in } E$$

where Q is an *observable* of a physical system under investigation and E is a Borel set on $\mathfrak{R}$. Here, we assume that the concept of an *observable of a physical system* is understood (an undefined concept) and that we have experimental procedures for determining the truth value of each statement of the above form. The statement takes the value 1 if it is *certainly true* and 0 if it is *not certainly true*.

In an axiomatic approach, we may consider U to be the set of all ordered pairs (Q, E), where Q is an observable and E a Borel set. Then it is assumed that there exist unambiguous experimental procedures for assigning 1 or 0, but not both, to each such ordered pair at any time.

Let L be the lattice of all the closed subspaces of a separable Hilbert space. Let ϑ be the set of all L-valued measures and let Γ be the set of all observables. We make the following assumptions, axioms for our empirical theory:

Q1: There is a bijection $h: \Gamma \rightarrow \vartheta$ and a surjection $f: U \rightarrow L$ such that $f(Q, E) = \bar{Q}(E)$, where $\bar{Q} = h(Q)$.

We note that the "first half" of this assumption corresponds to the well-known textbook axiom for quantum theory that the observables correspond one–one with self-adjoint operators. The "second half" of **Q1** corresponds to J. von Neumann's famous discovery that to every "quantum proposition" (Q, E) there corresponds a subspace of L.

If Q is an observable, then the corresponding L-valued measure $h(Q) = \bar{Q}$ shall also, where no confusion can arise, be denoted by Q. The next assumption states that nature allows only f-valuations, *where f is the function introduced in* **Q1**.

Q2: To every moment τ of time (i.e., to each $\tau \in \Re$) there exists an atom $e \in L$ such that the truth value (with regard to being certainly true or not) of any $p \in U$ at time τ is $t_e(p)$.

This merely says that at any time, the truth values of propositions (with regard to being certainly true or not) are determined by an f-valuation. One and only one valuation "prevails" at any time and it is an f-valuation. A law that, for a given physical system, determines which f-valuation prevails at which time can be postulated, but a study of the *dynamics* of f-valuations lies beyond the scope of this paper and shall be dealt with elsewhere.

If Ω is the set of all f-valuations and T is the theory of Ω, then T, as has already been said, is the set of all laws of nature governing the physical system under investigation. What the physicist calls *quantum theory* is, strictly speaking, not T itself, but (a subset of) *the metatheory of* T, i.e., a collection of properties of T and of its poset U_T. For example, axioms **Q1** and **Q2** belong to the metatheory of T. Gleason's (1957) theorem is not a logical consequence of T, but a statement *about* T. It says that the only measures on the orthoposet of T are those of a certain kind. Similarly, the statement that the allowed energy levels for the hydrogen atom are such and such is not a logical consequence of T, but *a property of* T. It says that the spectrum of the L-valued measure corresponding, by **Q1**, to the observable *energy* is such and such. This is a property of T, because L is isomorphic to U_T and every L-valued measure is, effectively, a U_T-valued measure.

With this distinction between T and quantum theory clearly understood, it nevertheless does no harm to identify T with quantum theory because, in a very colloquial sense, the properties of T *depend on* T and, hence may be considered to be consequences of T. Thus, quantum theory is a theory in a classical propositional calculus characterized by the fact that it has *properties* **Q1** and **Q2**. The next paragraph describes a property of T.

It is impossible for $C_Q(e)$ to be a singleton *for every* observable Q for any f-valuation t_e: If e is an atom, a one-dimensional subspace, then choose an orthonormal basis for the Hilbert space such that if $\psi \in e$ is a unit vector, then ψ is not an element of the chosen basis. Let Q be the observable whose Hermitian matrix is diagonal in this basis and has diagonal elements $a_1, a_2, a_3, \ldots$. Then $C_Q(e) = \{a_1, a_2, a_3, \ldots\}$, which is not a singleton. In an empirical theory constructed as above, uncertainty cannot be eliminated.

If t_e is an f-valuation, then e is a one-dimensional space generated by a unit vector ψ. Thus the f-valuation t_e is effectively determined by ψ and conversely. We call ψ *a state vector* of the system. As in quantum theory, each such ψ determines a *probability measure* on L and, therefore, on U_T (since U_T is isomorphic to L), i.e., a function $\rho_\psi : L \to [0, 1]$ such that

$$\rho_\psi(\mathbf{i}) = 1, \qquad \rho_\psi(\mathbf{o}) = 0$$

and

if $a_1, a_2, a_3, \ldots$ is a sequence of pairwise orthogonal elements in L, then

$$\textstyle\sum \rho_\psi(a_i) = \rho_\psi(\sum a_i)$$

where $\sum$ on the left denotes ordinary arithmetic sum of a series of real numbers and $\sum$ on the right denotes lattice sum. If $\mathbf{P}_a$ is the *projection operator* corresponding to the subspace a and ψ is a unit vector, then $\rho_\psi(a) = \langle \psi, \mathbf{P}_a \psi \rangle$, where $\langle \, , \rangle$ is the inner product in our Hilbert space. Every such probability measure determines a probability measure ρ_ψ^M on $\mathfrak{R}$ for each observable M as follows: For every Borel set E, put $\rho_\psi^M(E) = \rho_\psi(M(E))$. *Thus every f-valuation determines a probability measure on U_T and determines, for each observable, a probability measure on* $\mathfrak{R}$. Clearly, the subspace a contains the unit vector ψ iff $\rho_\psi(a) = 1$. Let e be the one-dimensional subspace containing ψ. It follows that, since $C_M(e)$ is the smallest Borel set E such that $M(E)$ contains e, then $\rho_\psi^M(C_M(e)) = 1$. A simple proposition which is certainly true, i.e., has truth value 1 in the f-valuation t_e, has probability 1 in that f-valuation. Put loosely, certainly true propositions have probability 1, as they should.

We note in passing that such an empirical theory need not necessarily be concerned with the microscopic world or even physics. An *observable* is simply any *quantitative* attribute of a system, i.e., an attribute that takes values in $\mathfrak{R}$. In economic affairs our physical system might be a market place in a fictitious country where, say, *the Bohr* is a currency unit, in honor of one of the great originators of quantum theory. (The reason why the country is fictitious shall soon appear.) "The price of a commodity" is, then, an observable; it has values in $\mathfrak{R}$. More accurately, each *commodity* is an observable. The *value* of the commodity is its price (per unit, e.g., kilogram or liter). "Apple," for instance, is one observable and "milk" is another.

The price of a commodity at any time, however, is fuzzy. It is not a point in $\mathfrak{R}$, but an extended set $C \subseteq \mathfrak{R}$, because it varies from one stall to another. C is the set of all available prices, of the commodity, in the market at a given time. The statement

"The price of apples, per kilo, is between 1 and 2 Bohrs"

is certainly true if the set C for apples is a subset of the open interval $(1, 2)$ and it is not certainly true otherwise, e.g., if C is partly in $(1, 2)$ and partly outside $(1, 2)$.

Let U be the set of all propositions of the form "*commodity M has a price in the Borel set E*" and suppose that in our fictitious country the number of commodities is so large that **Q1** can be satisfied and is in fact satisfied. (Thus the number of commodities is uncountable. This is why the country is fictitious.) If we also assume that "nature," or the nature of economic life in the country, allows only f-valuations so that also **Q2** is true, then the laws governing the market are embodied in the theory T of the set of all f-valuations. The poset of T is isomorphic to L, the lattice of all the closed subspaces of a separable Hilbert space. The simple propositions (or, more accurately, classes of such propositions), like quantum propositions, form a poset isomorphic to L. Each f-valuation determines what is certainly true at any time *and* the *probability* of each simple proposition. Formally, the theory T is indistinguishable from quantum theory.

Continuing our study of this fictitious market place, we note that the partial order relation in L reflects "causal implication" in the market in the following sense. Let p be the proposition

the price of apples is between 1 and 3 Bohrs

and q the proposition

the price of oranges is between 2 and 5 Bohrs

Then $(p \rightarrow q) \in T$ iff $f(p)$ is a subspace of $f(q)$ (Theorem 2). If indeed we have $f(p) \subseteq f(q)$, then we have the "causal" law that whenever the price of apples is *certainly* between 1 and 3 Bohrs, then the price of oranges is *certainly* between 2 and 5 Bohrs. If, on the other hand, we have that $f(p) \perp f(q)$, then we have the law that if the price of apples is *certainly* between 1 and 3 Bohrs, then the price of oranges is *not certainly* between 2 and 5 Bohrs (Theorem 5).

7. WAVE-PARTICLE DUALITY

This section is essentially conjectural: One of the main problems that led to the development of quantum theory and that still gives rise to controversy is that of wave-particle duality. How can a wave manifest itself as a set of particles? Quantum field theory (e.g., Tomonaga (1966) provides an adequate answer. The above approach to quantum theory seems to imply the elements of a new answer. Let our Hilbert space be the space $L^2(\mu)$ of all Lebesgue-measurable, square-integrable complex functions on ψ the x axis. Then ψ is a function of x, e.g., a function describing the dependence of the amplitude of a wave on x.

In quantum field theory such a wave can also be regarded as (or be associated with) a system of identical particles. Given any observable Q of such a particle, then, to put it somewhat imprecisely, ψ can be written as a linear combination of the eigenfunctions of Q. (This statement is an oversimplification, because an observable need not have eigenfunctions.) Given such an observable, then the question arises: Which points of the spectrum of Q are, or are available to be, occupied by particles in the state ψ? It seems reasonable to assume that the answer must be $C_Q(e)$, where e is the one-dimensional space generated by ψ. Independently of quantum field theory, but within the present approach, one may therefore associate with every observable Q and state ψ a set $X_Q(\psi)$ of *particles* such that $C_Q(e)$, with e being the one-dimensional space generated by ψ, is the set of points in the spectrum of Q where we may find a particle. In other words, in the state ψ, the observable Q has a value within $C_Q(e)$ *for every particle* in $X_Q(\psi)$. This is merely a conjecture at present and it needs to be developed further.

8. EPILOGUE ON THEORIES WITH ORTHOCOMPLEMENTATION

As I have already pointed out, the theory T is not a theory with orthocomplementation. In a theory with orthocomplementation the allowed valuations are determined not in terms of a single nonzero element of L, but in terms of two nonorthogonal atoms. It can be shown that any number of nonzero nonorthogonal elements can be used. For such a theory the converse of Theorem 5 is also true. As a result, occupation sets cannot be defined as in this paper. It seems that theories with orthocomplementation are related to richer quantum systems than those considered here. An analysis of such theories must be relegated to a future occasion.

REFERENCES

Gleason, A. M. (1957). *Journal of Mathematics and Mechanics*, **6**, 885.
Jauch, J. M. (1968). *Foundations of Quantum Mechanics*, Addison-Wesley.
Kochen, S., and Specker, E. (1967). *Journal of Mathematics and Mechanics*, **17**, 331.
Malhas, O. Q. (1987). *Journal of Symbolic Logic*, **52**(3), 834.
Tomonaga, S. (1966). *Quantum Mechanics*, Vol. II, North-Holland, Amsterdam.
Zierler, N., and Schlessinger, M. (1965). *Duke Mathematical Journal*, **32**, 251.